Some Random Famous Problems

Problem 1 (Zhou) You have 12 identical balls. One of the balls is heavier OR lighter than the rest (you don't know which). Using just a balance that can only show you which side of the tray is heavier, how can you determine which ball is the defective one? What is the fewest measurements needed? *Challenge: Can you generalize to n balls?*

Problem 2 (HMC Putnam Seminar)

a) Three points are chosen on a circle, and connected to form a triangle. What is the probability the center of the circle is contained in the interior of the triangle?

b) n points are randomly placed along the circumference of a circle. What is the probability that they all lie on the same semicircle?

Problem 3 (Drunk Passenger) A line of n airline passengers are waiting to board a plane. They each hold a ticket to one of the n seats. The first person in line is drunk and picks a random seat on the plane (all equally likely). The other passengers are sober and will go to their assigned seats unless it is already occupied; In that case, they will randomly choose a free seat. What is the probability that the last person in line sits in their assigned seat?

Problem 4 (The Egg Drop Problem) You are holding two eggs in a 100-story building. If an egg is thrown out of the window, it will not break if the the floor number is less than x, and it will always break if the floor number is equal to or greater than x. You would like to determine x. What is the strategy that will minimize the number of drops for the worst case scenario? *Challenge: Can you generalize for a m-story building with n-eggs*?

Problem 5 (Zhou) At a movie theater, a whimsical owner announces she will give a free ticket to the first person in line whose birthday is the same as someone who has already bought a ticket. You are given the opportunity to pick your position in line (it's a very long line). Assuming that you don't know anyone else's birthday and that all birthdays are distributed randomly throughout the year, what position in line gives you the largest chance at a free ticket? *Challenge: What is the expected value of the position where the first repeated birthday occurs?*

Problem 6 (Google?) There are 25 horses and a single racetrack. You can race 5 horses at a time but do not have a watch. What is the minimum number of races you need to identify the fastest 3 horses?

Hints:

1. Consider $n = 3$. How many weightings are needed? For general n, what if we split the balls into 3 equal groups?
2. Part a) may be related to part b)... For b), label the points and consider the events where each point is "leftmost" in it's semicircle.
3. Surprisingly, the answer does not depend on n. Try some small values and use induction.
4. if your first drop breaks on floor k, then how many drops must you use in the worst case now? What about if it doesn't break? Try to balance the "worst cases" for each of these scenarios. For the general case, try defining a function $f(m, n)$.
5. Suppose you are in position k, what is the probability that you are the first repeated birthday? Maximize this with some good ole calculus.
6. Experiment with fewer horses and see if you can find a strategy!