More of the Same

Problem 1) You start off with one amoeba. Every minute, this ameoba can either:

- Do nothing
- Die
- Split into two amoebas
- Split into three amoebas

Each of these actions has an equal probability of occurring. All further amoebas behave the exact same way. What is the probability that the amoebas eventually die off? *Challenge: Suppose now the amoeba has the ability to split into up to n amoebas. For large n, what is probability that the amoebas eventually die off now?*

Problem 2) A mouse is in the center of a circular pool of water with unit radius. Outside the pool lurks a vicious cat that wants to eat the mouse. The cat moves 4 times faster than the mouse can swim but the cat will not enter the water. Can the mouse find a way to escape the pool without the cat catching it the moment it reaches the edge of the pool? *Challenge: Say now the cat can move α times faster than the mouse can swim. What is the maximum α for which the mouse can escape?*

Problem 3) A $10 \times 10 \times 10$ cube made of 1000 unit cubes. How many cubes can you see on the outside?

Problem 4) At a movie theater, moviegoers line up to buy tickets. The ticket seller calls a patron **viewable** if he/she is taller than all of the people in front of him/her in line, otherwise the patron is **hidden**. Given that no two people are the exact same height, what is the expected number of viewable patrons?

Problem 5) n people each know a unique piece of gossip. When two of these people talk on the phone, they exchange all the items of gossip they know. What is the minimum number of calls the group must make to ensure everyone knows all gossip there is to know?

And now for a some variety!

Problem 6) Can you find $n \times n$ matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ such that

$$\boldsymbol{AB} - \boldsymbol{BA} = \boldsymbol{I}_n$$

where $\boldsymbol{I}_n$ is the $n \times n$ identity matrix?

Hints:

1. Let p be the probability a given amoeba and its descendants die out.
2. How long does it take the cat to do a lap around the pool? How long does it take the mouse to do a lap at radius $r \in (0, 1)$? Can we leverage this?
3. How many cubes are not on the outer layer?
4. Cough* linearity of expectation *cough
5. Experiment with small n and try to find optimal strategies. Drawing a graph could help?
6. Try to make it work for small n. Is there a relationship between the diagonal elements of $\boldsymbol{AB}$ and $\boldsymbol{BA}$? Is there a certain well known matrix tool that you can use?