# Semester Finale

**Problem 1) (Matic)** There are $M$ green apples and $N$ red apples in a basket. We take apples out randomly one by one until all the apples left in the basket are red. What is the probability that the moment we stop the basket is empty?

**Problem 2) (Matic)** A fair coin is tossed $n$ times. What is the expected product of the number of heads and the number of tails? *Challenge: What is the coin is biased and lands on heads with probability $p$.*

**Problem 3) (Matic)** You roll a fair $n$-sided die repeatedly and sum the outcomes. What is the expected number of rolls until the sum is a multiple of $n$ for the first time?

**Problem 4) (Stefanica)** The number $2^{29}$ has nine digits, all different. Without computing $2^{29}$, what is the missing digit?

**Problem 5)** When is the first time after 12pm that the minute and the hour hands meet again?

**Problem 6)** What is the value of $i^i$?

**Problem 7) (Amazon)** Let $S_5$ be the set of all $n = 5$ digit numbers that use each digit in $\{1, 2, 3, 4, 5\}$ exactly once. What is the sum all the numbers in $S_5$? *Challenge: What about for general $n \in [9]$?*

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And now for some variety!

**Problem 8) Thanks to Julius Barth for this neat problem!** There is a dragon terrorizing a village of $n$ people. Every day the dragon comes and eats one of the villagers until no more villagers are left. However, th villagers have poison pills available, and every day $i$ some number of villagers $x(i)$ will take a poison pill. If the dragon eats a poisoned villager the dragon will die along with the poisoned villagers on that day. If the dragon eats a non-poisoned villager, it survives, but all of the villagers die. How many villagers should take the poison pill each day to maximize the expected number of surviving villagers?

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Hints:

1. Consider the number of sequences where all apples are removed and a green apple is removed last. There is also a nice argument that involves finding disjoint events.
2. Given the number of heads, what can you say about the number of tails?
3. Consider the current sum modulo $n$.
4. Consider the number modulo 9
5. Use symmetry!
6. Is there another way to express $i$?
7. Try smaller $n$ and look for a pattern.
8. Try coding it and look for a pattern!