Mathematical Braintweasers

Problem 1) (Stefanica) Which is larger, e^π or π^e? *Challenge: Generalize to the case of* a^b *vs* b^a where $b > a > 0$.

Problem 2) (Stefanica) There is a row of 10 rooms and a treasure in one of them. Each night, a ghost moves the treasure to an adjacent room. You are trying to find the treasure but can only check one room per day. How do you find it? What is the minimum number of checks needed to guarantee you find the treasure? *Challenge: Generalize to the case of* n *rooms.*

Problem 3) (Stefanica) Given a set X with n elements, choose two subsets A and B at random. What is the probability that A is a subset of B?

Problem 4) (Stefanica) For every subset of $S = \{1, 2, 3, \ldots, 2013\}$ arrange the numbers in increasing order and take their sum with alternating signs. The resulting sum is called the *weight* of the subset. What is the total weight of all subsets in S? *Challenge: Generalize to the case of* $S = \{1, \ldots, n\}$.

Problem 5) (Stefanica) Select numbers uniformly distributed from $[0, 1]$ one after another as long as they keep decreasing; i.e. stop when you obtain a number greater than the previous one you selected.

a) On average, how many numbers have you selected when you stop?

b) What is the average value of the smallest number you have selected?

And now for some variety!

Problem 6) Show that there is no such number whose value doubles when its first digit is swapped with its last digit.

Hints:

1. $\log(1 + x) = x - x^2/2 + x^3/3 + \cdots$
2. If the treasure is in an even-numbered room to start...
3. Try conditioning on the size of B. You may find $(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ helpful.
4. Try some small values of n. Is there a pattern? For proof, the binomial theorem may help.
5. For a positive valued discrete random variable N we have $\mathbb{E}[N] = \sum_{n=0}^{\infty} \Pr[N > n]$. For b), begin by conditioning on the turn we stop.