Probably Probability

Problem 1) (Matic) You toss a fair coin n times. What is the expected product of the number of heads and the number of tails?

Problem 2) (Matic) An unfair coin is tossed n times and all n tosses result in heads. If the coin is tossed one more time, what is the probability that it results in heads? (Assume the probability of heads p, is a random variable uniformly distributed on $[0, 1]$.)

Problem 3) (Matic) Consider 2^n players of equal skill playing a game where the players are paired off against each other at random. The 2^{n-1} winners are again paired off randomly, and so on, until only a single winner remains. If A and B are players in this game, what is the probability that they never play each other?

Problem 4) (Gelca) Let α and β be given positive real numbers, with $\alpha < \beta$. If two points are selected at random from a straight line segment of length β, what is the probability that the distance between them is at least α?

Problem 5) (MIT Putnam Seminar) A deck of cards (with 26 red cards and 26 black cards) is shuffled, and the cards are turned face up one at a time. At any point during this process, before the last card is turned up, you can stay "stop." If the next card is red, you win \$1; if it is black, you win nothing. What is your best strategy? In particular, is there a strategy that gives you an expectation of better than 50 cents?

And now for some variety!

Problem 6) 3D Printed Statues You have a single 3D printer and would like to use it to produce statues. However, printing the statues one by one on the 3D printer takes a long time, so it may be more time-efficient first to use the 3D printer to print a new printer. That new printer may then be used to print statues or even more printers. Print jobs take a full day, and every day you can choose for each printer in your possession to have it print a statue or to have it 3D print a new printer (which becomes available for use the next day). What is the minimum possible number of days needed to print at least n statues?

Hints:

1. If you have H heads, how many tails must you have? How could you leverage the closed forms for the variance and expectation for a binomial RV?
2. Bayes Theorem may help here
3. What would need to happen for A and B to meet in round i for $i = 1, \ldots, n$?
4. Set up an integral!
5. Experiment with some smaller (even number of cards) decks first. Does anything help you do better than 50 cents?

Solution: Problem 1

Let H and T be random variables denoting the number of heads/tails in the n throws. Suppose for now the probability of heads on any given throw is p. The number of tails is given by $T = n - H$. If we now compute the expected product, we find

$$\mathbb{E}[H \cdot T] = \mathbb{E}[H \cdot (n - H)] = n\mathbb{E}[H] - \mathbb{E}[H^2]$$

Note that $H \sim Bin(n, p)$ and thus $\mathbb{E}[H] = np$ and $Var(H) = np(1-p)$. If we recall the standard formula for variance $Var(H) = \mathbb{E}[H^2] - \mathbb{E}[H]^2$, we can write

$$\begin{aligned} \mathbb{E}[H \cdot T] &= n\mathbb{E}[H] - \mathbb{E}[H^2] \\ &= n(np) - \left(Var(H) + \mathbb{E}[H]^2\right) \\ &= n^2p - (np(1-p) + (np)^2) \\ &= n(n-1)p(1-p) \end{aligned} \tag{1}$$

plugging in $p = 1/2$ yields our solution $\boxed{n(n-1)/4}$/

Solution: Problem 2

Let A be the event that the next throw is heads, B be the event that the first n throw was all heads and P be a random variable that is the true probability of heads on the coin. We would like to evaluate $\Pr(A|B)$. We can do so by further conditioning on P.

$$\Pr(A|B) = \int_0^1 \Pr(A|B, P = p) \cdot f(p|B)dp \tag{2}$$

where $f(p|B)$ is the probability density function of p given B. To evaluate this, we must first evaluate each of the two terms in the integrand. Starting with $\Pr(A|B, P = p)$. Note that if we know the 'true' probability of the coin p then the next toss is independent of the previous tosses and thus $\Pr(A|B, P = p) = p$. As for the second term, we have via Bayes Rule

$$\begin{aligned} f(p|B) = \frac{\Pr(B|P = p) \cdot f(p)}{\Pr(B)} & \\ &= \frac{\Pr(B|P = p) \cdot f(p)}{\int_0^1 \Pr(B|P = q) \cdot f(q)} \\ &= \frac{p^n \cdot dp}{\int_0^1 q^n \cdot dq} \\ &= (n+1)p^n dp. \end{aligned}$$

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Putting these two together, we find

$$\begin{aligned}
\Pr(A|B) &= \int_0^1 \Pr(A|B, P = p) \cdot f(p|B) \\
&= \int_0^1 p \cdot (n+1)p^n dp \\
&= (n+1) \int_0^1 p^{n+1} dp \\
&= \boxed{\frac{n+1}{n+2}}.
\end{aligned} \tag{3}$$

Note this result matches our intuition in the case of $n = 0$ (no previous throws) and in the limit $n \to \infty$.

Solution: Problem 3

In this case, it is easier to find the complement. $\Pr(A \text{ doesn't play } B) = 1 - \Pr(A \text{ plays } B)$. Using disjoint cases, we have

$$\Pr(A \text{ plays } B) = \sum_{k=1}^{n} \Pr(A \text{ plays } B \text{ in round } k). \tag{4}$$

To find $\Pr(A \text{ plays } B \text{ in round } k)$, we note that several things have to happen:

1. For A and B to meet in the k-th round, they must both be in the same size 2^k sub-tournament.
2. Given that they are in the size 2^k sub-tournament, they must be on opposite sides of the bracket.
3. They both must win their $k-1$ games to reach the k-th round.

The probability of each of these events happening is

$$\Pr(A \text{ plays } B \text{ in round } k) = \frac{2^k - 1}{2^n - 1} \cdot \frac{2^{k-1}}{2^k - 1} \cdot \left(\frac{1}{2}\right)^{2(k-1)} = \frac{(\frac{1}{2})^{k-1}}{2^n - 1}.$$

Now if we sum up the k, we find

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$$\begin{aligned}\Pr(A \text{ plays } B) &= \sum_{k=1}^{n} \frac{(\frac{1}{2})^{k-1}}{2^n - 1} \\ &= \frac{1}{2^n - 1} \sum_{k=1}^{n} \left(\frac{1}{2}\right)^{k-1} \\ &= \frac{1}{2^{n-1}}\end{aligned} \tag{5}$$

and thus $\boxed{\Pr(A \text{ doesn't play } B) = 1 - \frac{1}{2^{n-1}}}$

Solution: Problem 4

Solution: Problem 5

Solution: Problem 6

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