This Week's Problems

Problem 1) Pig In the game of Pig, two players take turns rolling a die. On a turn, a player may roll the die as many times as they like, provided they have not thrown a one. If they end their turn before rolling a one, their turn score is the sum of rolls for that turn. If they roll a one, their turn score is zero. At the end of the turn, their turn score is added to the player's total score. The first player to reach 100 points wins.

1. Let's consider the strategy for playing this game in which the player will roll until their turn score is at least M. What value of M will maximize their expected turn score? What is the expected value?

2. Suppose in a game of Pig, a player decides to just go for it and try to roll 100 points on their first turn. What is the probability that they will succeed?

Problem 2) Suppose we roll n dice, remove all the dice that come up 1, and roll the rest again. If we repeat this process, eventually all the dice will be eliminated. How many rolls, on average, will we make? Show, for instance, that on average fewer than $O(\log n)$ throws occur.

Problem 3) Chess Tournament A chess tournament has 128 players, each with a distinct rating. Assume that the player with the higher rating always wins against a lower rated opponent and that the winner proceeds to the subsequent round. What is the probability that the highest rated and second-highest rated players will meet in the final? *Challenge: Generalize it to a tournament with 2^n players.*

Problem 4) Place or take You are playing a one-player game with two opaque boxes. At each turn, you can choose to either "place" or "take". "Place" places \$1 from a third party into one box randomly. "Take" empties out one box randomly and that money is yours. This game consists of $N = 100$ turns where you must either place or take. Assuming optimal play, what is the expected payoff of this game? Note that you do not know how much money you have taken until the end of the game.

Hints:

1. What does convex mean? What is a nice property of expectation?
2. Consider working backwards! What should we do with one candidate left? Two? After each interview we make a decision, can you write down a relationship for the value of each decision? When do we pick either one?
3. Try to set up an equation for the probability of winning for a given n and p. Optimize baby!
4. A useful notation may be to introduce p_1 as the probability the ant escapes given 1 step from the edge. How could we use this? How does it relate to p_2 ? p_3? (maybe something recursive?)